Section 3.1 CONVERGENCE

Definition 1 *A* ***sequence*** *of real numbers (or a sequence in* $\mathbb{R}$*) is a function whose domain is* $\mathbb{N}$ *and whose range is a subset of* $\mathbb{R}$*.*

If x is a sequence in $\mathbb{R} \Rightarrow x : \mathbb{N} \to \mathbb{R}$ with $x(n) = x_n$ for each $n \in \mathbb{N}$. We denote it by

$$x = (x_n)_{n\in\mathbb{N}} = (x_n)_{n=1}^{\infty} = (x_1, x_2, x_3, ...) = (x_n).$$

Example 2 *The sequence* $(\frac{(-1)^n}{n})_{n=1}^{\infty}=$

Definition 3 *A sequence* $(x_n)_{n\in\mathbb{N}}$ ***eventually*** *has a certain property if* $\exists n_0 \in \mathbb{N}$ *s.t.*

$$(x_n)_{n\geq n_0} =$$

has this property.

Example 4 *The sequence* $(100, -5, \pi, 1, 0, 6, 13, 2, 4, 8, 16, 32, 64, ...)$ *is eventually what? Why?*

Definition 5 *For* $x \in \mathbb{R}$ *and* $\varepsilon > 0$*, the open interval*

$$(x - \varepsilon, x + \varepsilon) = \{y \in \mathbb{R} : |y - x| < \varepsilon\}$$

center at x of radius ε is a $(\varepsilon-)$ **neighborhood** of x.

Example 6 *Let* $x = 1$*, sketch the neighborhood of* x *in* $\mathbb{R}$ *for* $\varepsilon = .5$ *and* $\varepsilon = 2$.

(Sec 3.1 cont.)

In Calculus, we learned that $\lim_{n\to\infty} \frac{1}{n} = 0$ (i.e., the sequence $(\frac{1}{n})_{n\in\mathbb{N}}$ converges to 0). How does the nbh. of 0 relate to the terms of the sequence $(\frac{1}{n})_{n\in\mathbb{N}}$?

Definition 7 *The sequence $(x_n)_{n\in\mathbb{N}}$* ***converges*** *to a real number x (or has limit x) if $\forall$nbh. U of x, the sequence $(x_n)_{n\in\mathbb{N}}$ is eventually in U.*

Definition 8 *(a proposition that is often viewed as $\langle Def \rangle$)*

A sequence $(x_n)_{n\in\mathbb{N}}$ is said to converge to $x \in \mathbb{R} \iff$

$$\begin{aligned} \forall\varepsilon &> 0\ \exists n_0 \in \mathbb{R} \text{ s.t. for all } n \in \mathbb{N},\ n > n_0 \Rightarrow |x_n - x| < \varepsilon, \\ \text{or } \forall\varepsilon &> 0\ \exists n_0 \in \mathbb{N} \text{ s.t. for all } n \in \mathbb{N},\ n \geq n_0 \Rightarrow |x_n - x| < \varepsilon. \end{aligned}$$

Notation 9 *If $(x_n)_{n\in\mathbb{N}}$ converges to x, then x is called the* ***limit*** *of the seq. $(x_n)_{n\in\mathbb{N}}$, and we write*

$$\lim_{n\to\infty} x_n = x,\ \ \lim x_n = x,\ \ \text{or } x_n \to x.$$

Definition 10 *A seq. that is not convergent is* ***divergent.***

(Sec 3.1 cont.)

Problem 11 *Prove that*

$$\lim_{n\to\infty} \frac{1}{n^3} = 0.$$

$\langle Pf \rangle$:

(Sec 3.1 cont.)

Problem 12 *Prove that*

$$\lim_{n\to\infty} \frac{n^2+1}{2n^2+5} = \frac{1}{2}.$$

$\langle Pf \rangle$:

(Sec 3.1 cont.)

Exercise 13 *Given distinct points in* $\mathbb{R}$*, can we always separate them by* ***disjoint*** *nbh.s?*

Lemma 14 *If* $x, y \in \mathbb{R}$ *with* $x \neq y$

$\Longrightarrow \exists$ neighborhoods U of x and V of y s.t. $U \cap V = \varnothing$.

$\langle Pf \rangle$: visualize your strategy first:

(Sec 3.1 cont.)

Theorem 15 *If a sequence converges*

$\Longrightarrow$its limit is unique.

$\langle Pf \rangle$: visualize your strategy first:

(Sec 3.1 cont.)

Limit Does Not Always Exist (Review your notes in MATH 355)

$\langle Note \rangle$ The sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ does not converge
$\iff \lim x_n \neq x$ for all $x \in \mathbb{R}$

$\lim x_n \neq x$
$\iff \exists$nbh. U of x s.t. the sequence (x_n) is not eventually in U
$\iff \exists$nbh. U of x s.t. $\forall n_0 \in \mathbb{N}, \exists n \geq n_0$ s.t. $x_n \notin U$
$\iff \exists$nbh. U of x s.t the sequence $(x_n)_{n\in\mathbb{N}}$ is **frequently outside** of U.
$\iff$(the negation of Definition 8) $\exists \varepsilon > 0$ s.t. $\forall n_0 \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq n_0$ with $|x_n - x| \geq \varepsilon$

Read Other Techniques for Convergence in p.42-43.

Problem 16 *Prove: The sequence* $(x_n) = ((-1)^n)$ *is divergent.*